Mixing angles of quarks and leptons as an outcome of $SU(2)$ horizontal symmetries

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- Q. Duret & B. Machet, Phys. Lett. B 643 (2006) 303-310, arXiv :hep-ph/0606303.
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Introduction

Non-degenerate coupled systems :

- Bosons, such as binary systems of neutral kaons $K_L - K_S$
- Fermions in the standard model (coupled through non-diagonal Yukawa couplings in flavour space)

A fundamental feature :

In QFT, due to the mass differences between particles, mixing matrices of such systems should *a priori* never be considered as unitary

$\Longrightarrow$ In the following approach, considering massive fermions in the SM :

- We parametrize mixing matrices as non-unitary
- We get the maximal mixing through basic physical requirements, equivalent to imposing *a posteriori* the unitarity.
- We show that the mixing angles of quarks and leptons satisfy conditions corresponding to a special pattern of unitarity violation related to $SU(2)$ horizontal symmetries.

Flavour and mass states in QFT

Two types of states :

Flavour eigenstates		Mass eigenstates
$(e_f^-, \mu_f^-, \nu_{e,f}, \nu_{\mu,f} \ldots)$	**VS**	$(e_m^-, \mu_m^-, \nu_{e,m}, \nu_{\mu,m} \ldots)$
=		=
gauge interaction eigenstates		propagating eigenstates

In QFT the physical masses are the poles of the full renormalized propagator, i.e. the values of $z = q^2$ which satisfy

$$\det \Delta^{-1}(z) = 0, \text{ for } z = z_i, \tag{1}$$

The mass eigenstates are the corresponding eigenvectors :

$$\Delta^{-1}(z = z_i)\varphi_m^i = 0. \tag{2}$$

In terms of the renormalized quadratic lagrangian $L^{(2)}(z) = \Delta^{-1}(z)$:

$$\det L^{(2)}(z) = 0 \qquad\qquad L^{(2)}(z = z_i)\varphi_m^i = 0. \tag{3}$$

Why mixing matrices have no reasons to be unitary

The mixing matrices connect flavour eigenstates (Ψ_f) to mass eigenstates (Ψ_m) :

$$\Psi_f = K\Psi_m. \tag{4}$$

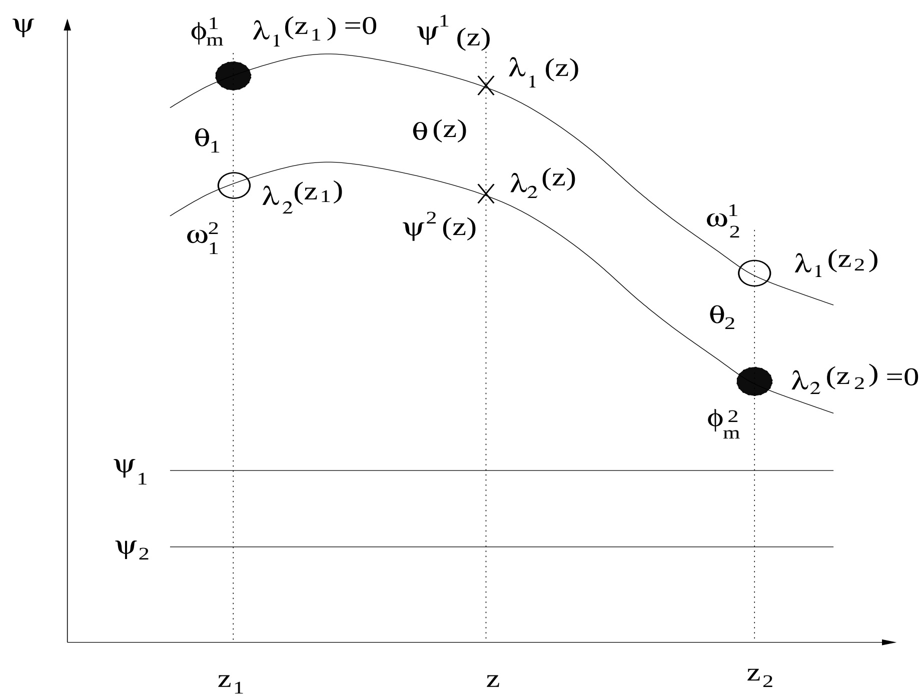
$L^{(2)}(z = q^2)$ hermitian

$\Longrightarrow$ at each z, the eigenstates of $L^{(2)}(z)$ form an orthonormal basis $\Psi(z)$.

The mass eigenstates respectively belong to different orthonormal bases

$\Longrightarrow$ they do no form themselves an orthonormal basis

If the flavour states form an orthonormal basis,
the mixing matrix K cannot be unitary.



Leptonic weak neutral currents : 2 generations

$$\Psi_f = J\Psi_m, \quad \Psi_f = \begin{pmatrix} \nu_{e,f} \\ \nu_{\mu,f} \\ e^-_f \\ \mu^-_f \end{pmatrix}, \quad \Psi_m = \begin{pmatrix} \nu_{e,m} \\ \nu_{\mu,m} \\ e^-_m \\ \mu^-_m \end{pmatrix}, \quad J = \left(\begin{array}{c|c} K_\nu & \\ \hline & K_\ell \end{array}\right), \tag{5}$$

- Parametrize the mixing matrices as non-unitary with two angles instead of one (preserving a unit norm for all states) :

$$K_\nu = \begin{pmatrix} e^{i\alpha}c_1 & e^{i\delta}s_1 \\ -e^{i\beta}s_2 & e^{i\gamma}c_2 \end{pmatrix}, \qquad K_\ell = \begin{pmatrix} e^{i\theta}c_3 & e^{i\zeta}s_3 \\ -e^{i\chi}s_4 & e^{i\omega}c_4 \end{pmatrix}. \tag{6}$$

- Neutral currents $(W^3_\mu) \quad \Longrightarrow \quad K^\dagger_\nu K_\nu,\ K^\dagger_l K_l$

- Two characteristics in flavour space : universality and absence of FCNC.

– If K unitary $\quad\Longrightarrow\quad$ automatically achieved in mass space too.
– If K non-unitary $\quad\Longrightarrow\quad$ no longer automatic.

Hence we impose **I** : universality of neutral currents, and **II** : absence of ”FCNC” in the space of mass eigenstates (experimentally observed).

Leptonic weak neutral currents : 2 generations

Given

$$K_\nu^\dagger K_\nu = \begin{pmatrix} c_1^2 + s_2^2 & c_1 s_1 e^{i(\delta-\alpha)} - c_2 s_2 e^{i(\gamma-\beta)} \\ c_1 s_1 e^{i(\alpha-\delta)} - c_2 s_2 e^{i(\beta-\gamma)} & s_1^2 + c_2^2 \end{pmatrix} \tag{7}$$

these constraints translate into :

– **I** : identity of diagonal elements : $c_1^2 + s_2^2 = c_2^2 + s_1^2$

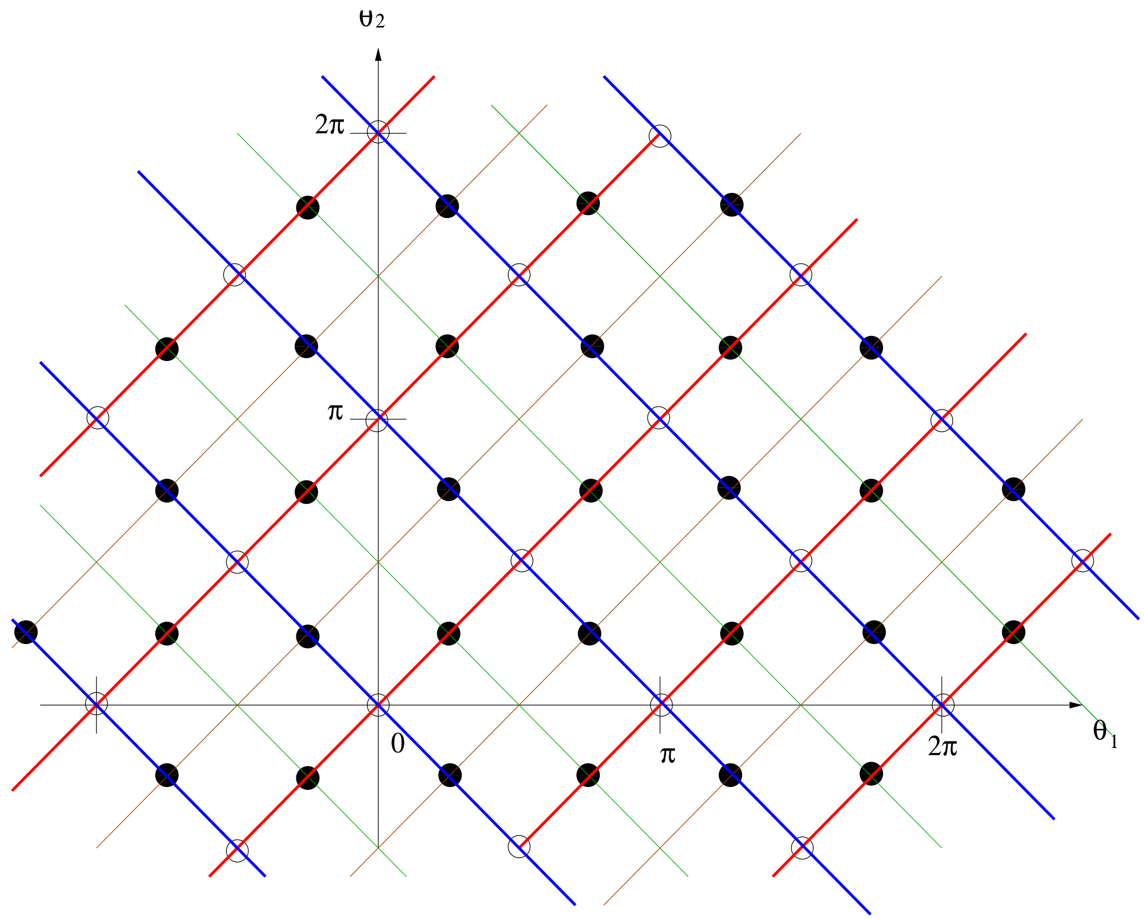
– **II** : vanishing of non-diagonal elements : $c_1 s_1 = c_2 s_2$ or $c_1 s_1 = -c_2 s_2$.

Two sets of solutions arise :

- One-parameter ("Cabibbo-like") solutions : $\theta_2 = \pm\theta_1 + k\pi$ for which **I** and **II** coincide.

- Two-parameter solutions, for which **I** and **II** are independent.
  They are of the form $\theta_1 = \pm\frac{\pi}{4}$; $\theta_2 = \pm\theta_1 + k\pi$, i.e. give rise to maximal mixing.

Mixing angles



Constraints given by the two conditions of universality and absence of FCNC's.

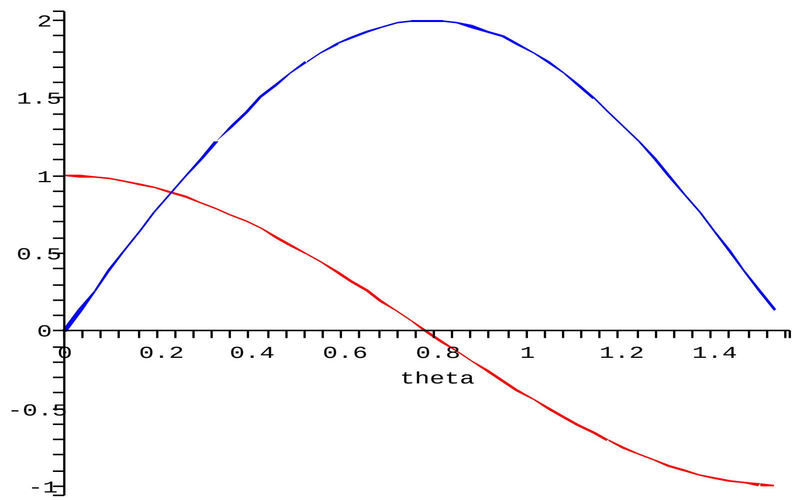
Getting the Cabibbo angle

Neighborhood of the Cabibbo case : $\theta_2 = \pm\theta_1 + \epsilon$.

$$\Longrightarrow \quad K \text{ deviates from unitarity by } K^\dagger K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} \sin(2\theta_c) & -a\cos(2\theta_c) \\ -a^*\cos(2\theta_c) & -\sin(2\theta_c) \end{pmatrix}.$$

- Conditions **I** and **II** cannot any more be simultaneously fulfilled

- But **I** and **II** reduce to a single condition for a value of θ_c which turns out to be that of the Cabibbo angle experimentally measured.



$$\tan(2\theta_c) = \frac{1}{2} \quad \Longrightarrow \quad \cos\theta_c = 0.9732.$$

An interpretation : unitarity violation in terms of an $SU(2)$ flavour symmetry

Consider for example the (d,s) channel.

Departure from unitarity ensures :

$$\mathcal{L} \propto W_\mu^3 \left[\alpha\, \bar{d}_m \gamma_L^\mu d_m + \beta\, \bar{s}_m \gamma_L^\mu s_m + \delta\, \bar{d}_m \gamma_L^\mu s_m + \zeta\, \bar{s}_m \gamma_L^\mu d_m\right], \quad \alpha \neq \beta, \quad \delta \neq 0, \quad \zeta \neq 0. \tag{8}$$

Imposing the condition that

– the universality of $\bar{d}_m \gamma_L^\mu d_m$ and $\bar{s}_m \gamma_L^\mu s_m$ currents

– the absence of $\bar{d}_m \gamma_L^\mu s_m$ and $\bar{s}_m \gamma_L^\mu d_m$ currents

are violated with the same strength, leads to

$$\delta = \alpha - \beta = \zeta. \tag{9}$$

The Lagrangian then takes the form

$$\mathcal{L} \propto W_\mu^3 \left[(\alpha+\beta)\frac{\bar{d}_m \gamma_L^\mu d_m + \bar{s}_m \gamma_L^\mu s_m}{2} + (\alpha-\beta)\left(\frac{\bar{d}_m \gamma_L^\mu d_m - \bar{s}_m \gamma_L^\mu s_m}{2} + \bar{d}_m \gamma_L^\mu s_m + \bar{s}_m \gamma_L^\mu d_m\right)\right], \tag{10}$$

which makes appear an underlying ”horizontal” $SU(2)$ symmetry relating d and s.

Going to three generations

• General parametrization :

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -\tilde{s_{23}} & \tilde{c_{23}} \end{pmatrix} \times \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -\tilde{s_{13}} & 0 & \tilde{c_{13}} \end{pmatrix} \times \begin{pmatrix} c_{12} & s_{12} & 0 \\ -\tilde{s_{12}} & \tilde{c_{12}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11}$$

$*$ absence of $\{13\}$ and $\{31\}$ currents :

$$c_{12}\left[c_{13}s_{13} - \tilde{c}_{13}\tilde{s}_{13}(\tilde{c}^2_{23} + s^2_{23})\right] - \tilde{c}_{13}\tilde{s}_{12}(c_{23}s_{23} - \tilde{c}_{23}\tilde{s}_{23}) = 0; \tag{12}$$

$*$ absence of $\{23\}$ and $\{32\}$ currents :

$$s_{12}\left[c_{13}s_{13} - \tilde{c}_{13}\tilde{s}_{13}(\tilde{c}^2_{23} + s^2_{23})\right] + \tilde{c}_{13}\tilde{c}_{12}(c_{23}s_{23} - \tilde{c}_{23}\tilde{s}_{23}) = 0; \tag{13}$$

$*$ absence of $\{12\}$ and $\{21\}$ currents :

$$s_{12}c_{12}c^2_{13} - \tilde{s}_{12}\tilde{c}_{12}(c^2_{23} + \tilde{s}^2_{23}) + s_{12}c_{12}\tilde{s}^2_{13}(s^2_{23} + \tilde{c}^2_{23}) + \tilde{s}_{13}(s_{12}\tilde{s}_{12} - c_{12}\tilde{c}_{12})(c_{23}s_{23} - \tilde{c}_{23}\tilde{s}_{23}) = 0; \tag{14}$$

$*$ equality of $\{11\}$ and $\{22\}$ currents :

$$(c^2_{12} - s^2_{12})\left[c^2_{13} + \tilde{s}^2_{13}(s^2_{23} + \tilde{c}^2_{23})\right] - (\tilde{c}^2_{12} - \tilde{s}^2_{12})(c^2_{23} + \tilde{s}^2_{23}) + 2\tilde{s}_{13}(c_{23}s_{23} - \tilde{c}_{23}\tilde{s}_{23})(c_{12}\tilde{s}_{12} + s_{12}\tilde{c}_{12}) = 0; \tag{15}$$

$*$ equality of $\{22\}$ and $\{33\}$ currents :

$$s^2_{12} + \tilde{c}^2_{12}(c^2_{23} + \tilde{s}^2_{23}) - (s^2_{23} + \tilde{c}^2_{23}) + (1 + s^2_{12})\left[\tilde{s}^2_{13}(s^2_{23} + \tilde{c}^2_{23}) - s^2_{13}\right] + 2s_{12}\tilde{s}_{13}\tilde{c}_{12}(\tilde{c}_{23}\tilde{s}_{23} - c_{23}s_{23}) = 0. \tag{16}$$

$$\theta_{13} = 0 = \tilde{\theta}_{13}$$

One has the same type of solutions as for two generations :

θ_{12} and θ_{23} may be

- Cabibbo-like ($\tilde{\theta}_{12,23} = \theta_{12,23} + k\pi$)
- or maximal ($\theta_{12,23} = \pi/4 + n\pi/2,\ \tilde{\theta}_{12,23} = \pi/4 + m\pi/2$)

- θ_{12} and θ_{23} both Cabibbo-like

$$\Longrightarrow \text{Quarks mixing pattern}$$

- θ_{12} Cabibbo-like and θ_{23} maximal

$$\Longrightarrow \text{Neutrinos mixing pattern}$$

Leptonic θ_{12} and Quark-Lepton Complementarity

We consider the case

$$\begin{cases} \theta_{13} = 0 = \tilde{\theta}_{13} \\ \tilde{\theta}_{12} = \theta_{12} + \epsilon \\ \tilde{\theta}_{23} = \theta_{23} + \eta \quad ; \quad \theta_{23} = \frac{\pi}{4} \end{cases} \tag{17}$$

The absence of {1,2} {2,1} non-diagonal neutral currents is violated with the same strength as the universality of {11} {22} neutral currents for

$$-\eta s_{12}c_{12} + \epsilon(s_{12}^2 - c_{12}^2)(1+\eta) = -\eta(c_{12}^2 - s_{12}^2) + 4\epsilon s_{12}c_{12}(1+\eta) \tag{18}$$

i.e., neglecting the terms proportional to ϵ :

$$\boxed{\tan(2\theta_{12}) = 2 \qquad \Longrightarrow \qquad \theta_{12}^{\ell} = 31.7}$$

One immediately obtains

$$\tan(2\theta_c) = \frac{1}{\tan 2\theta_{12}^{\ell}} \tag{19}$$

whence

$$\boxed{\theta_c + \theta_{12}^{\ell} = \frac{\pi}{4} \qquad \Longrightarrow \qquad \text{Quark-Lepton} \quad \text{Complementarity}}$$

3 generations : general resolution

$$\begin{cases} \tilde{\theta}_{12} = \theta_{12} + \epsilon \\ \tilde{\theta}_{23} = \theta_{23} + \eta \\ \tilde{\theta}_{13} = \theta_{13} + \rho \end{cases} \tag{20}$$

The l.h.s.'s of the five preceeding equations depart from zero by

$$\eta c_{13}\left[s_{12}(c_{23}^2 - s_{23}^2) + 2s_{13}c_{12}c_{23}s_{23}\right] - \rho c_{12}(c_{13}^2 - s_{13}^2); \tag{21a}$$

$$\eta c_{13}\left[-c_{12}(c_{23}^2 - s_{23}^2) + 2s_{13}s_{12}c_{23}s_{23}\right] - \rho s_{12}(c_{13}^2 - s_{13}^2); \tag{21b}$$

$$-\epsilon(c_{12}^2 - s_{12}^2) + \eta\left[s_{13}(c_{23}^2 - s_{23}^2)(c_{12}^2 - s_{12}^2) - 2c_{23}s_{23}c_{12}s_{12}(1 + s_{13}^2)\right] + 2\rho c_{13}s_{13}c_{12}s_{12}; \tag{21c}$$

$$4\epsilon c_{12}s_{12} + \eta\left[-4s_{13}s_{12}c_{12}(c_{23}^2 - s_{23}^2) - 2c_{23}s_{23}(c_{12}^2 - s_{12}^2)(1 + s_{13}^2)\right] + 2\rho c_{13}s_{13}(c_{12}^2 - s_{12}^2); \tag{21d}$$

$$-2\epsilon s_{12}c_{12} + \eta\left[2s_{13}c_{12}s_{12}(c_{23}^2 - s_{23}^2) + 2c_{23}s_{23}\left((c_{12}^2 - s_{12}^2) + c_{13}^2(1 + s_{12}^2)\right)\right] + 2\rho c_{13}s_{13}(1 + s_{12}^2); \tag{21e}$$

$$2\epsilon s_{12}c_{12} + \eta\left[-2s_{13}c_{12}s_{12}(c_{23}^2 - s_{23}^2) + 2c_{23}s_{23}\left(c_{13}^2(1 + c_{12}^2) - (c_{12}^2 - s_{12}^2)\right)\right] + 2\rho c_{13}s_{13}(1 + c_{12}^2). \tag{21f}$$

3 generations : general resolution

- Resolution by substitution of the three non-vanishing parameters ϵ, η and ρ.

- 8 possibilites due to the three choices of sign in the conditions :

| violation of universality | = | violation of the absence of FCNC's |.

$\Longrightarrow$ One is led to

$$F(\theta_{12}, \theta_{23}, \theta_{13}) = 0 \tag{22}$$

which is solved numerically in θ_{13} while assigning to θ_{12} and θ_{23} their values already predicted (resp. $\frac{1}{2}\arctan(2)$ and $\frac{\pi}{4}$).

$\Longrightarrow$ One obtains among all the solutions :

$$\theta_{13} = \pm 0.2717 \quad , \quad \sin^2(\theta_{13}) = 0.072,$$

$$\theta_{13} = \pm 5.7\,10^{-3} \quad , \quad \sin^2(\theta_{13}) = 3.3\,10^{-5}. \tag{23}$$

Conclusion

- The unitarity of the mixing matrices may be considered in the SM as an accidental characteristic.

- The mixing angles measured at present for neutrinos (as well as quarks) are found in the neighbourhood of this standard situation ; they are obtained through conditions corresponding to a departure from unitarity that arise due to mass splittings.

- However these mixing angles are obtained in a way that is independent of any fermionic mass spectrum.

- This pattern of ”unitarity breaking” is linked to a series of $SU(2)$ horizontal symmetries linking each pair of fermions of the same isospin.

- Our approach also yields an exact realization of the Quark-Lepton Complementarity

- Links with the other approaches (esp. tri-bimaximal mixing) ?